

MTH250 - CALCULUS

Lecture No. 19

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These lecture notes are prepared to serve as handouts for the video lectures on the course *MTH-250 Calculus* of Virtual Campus, COMSATS Institute of Information Technology, Islamabad, Pakistan. The notes were derived from an extensive collection of notes and books. Most of the theoretical details as well as worked problems are from *Calculus*, by *H. Anton, I Bivens and S. Davis*, John Wiley & Sons, 10th edition, 2012.

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LECTURE 19

Integration III

19.1 Substitution for definite integrals

Indefinite integrals of the form $\int f(g(x))g'(x)dx$ can be evaluated by making the substitution $u = g(x)$ and using $du = g'(x)dx$. In order to apply this method to definite integrals of the form $\int_a^b f(g(x))g'(x)dx$, we need to account for the effect that the substitution has on the x -limits of integration. There are two methods.

1. Evaluate the indefinite integral $\int f(g(x))g'(x)dx$. Then use

$$\int_a^b f(g(x))g'(x)dx = \left[\int f(g(x))g'(x)dx \right]_a^b.$$

This procedure does not require any modification of the x -limits of integration.

2. Make the substitution $u = g(x)$, this implies $du = g'(x)dx$. Then replace the x -limits $x = a$ and $x = b$ by u -limits

$$u = g(a) \quad u = g(b),$$

This produces a new definite integral

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

Finally, solve the new integral.

Example 19.1. Evaluate $\int_0^2 x(x^2 + 1)^3 dx$.

Solution by Method 1. If we let $u = x^2 + 1$ so that $du = 2x dx$ then we obtain

$$\int x(x^2 + 1)^3 dx = \frac{1}{2} \int u^3 du = \frac{u^4}{8} + C = \frac{(x^2 + 1)^4}{8} + C$$

Thus

$$\int_0^2 x(x^2 + 1)^3 dx = \left[\int x(x^2 + 1)^3 dx \right]_0^2 = \frac{(x^2 + 1)^4}{8} \Big|_0^2 = \frac{625}{8} - \frac{1}{8} = 78.$$

Solution by Method 2. If we make the substitution $u = x^2 + 1$, then

- if $x = 0$, $u = 1$,
- if $x = 2$, $u = 5$

Thus

$$\int_0^2 x(x^2 + 1)^3 dx = \frac{1}{3} \int_1^5 u^3 du = \frac{u^4}{8} \Big|_{u=1}^{u=5} = \frac{625}{8} - \frac{1}{8} = 78.$$

which agrees with the result obtained by Method 1. □

Theorem 19.1. If g' is continuous on $[a, b]$ and f is continuous on an interval containing the values of $g(x)$ for $a \leq x \leq b$, then

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du.$$

Proof. Since f is continuous on an interval containing the values of $g(x)$ for $a \leq x \leq b$, it follows that f has an anti-derivative F on that interval. If we let $u = g(x)$, then the chain rule implies that

$$\frac{d}{dx} F(g(x)) = \frac{d}{dx} F(u) = \frac{dF}{du} \frac{du}{dx} = f(u) \frac{du}{dx} = f(g(x))g'(x)$$

for each x in $[a, b]$. Thus, $F(g(x))$ is an anti-derivative of $f(g(x))g'(x)$ on $[a, b]$. Therefore, by the fundamental theorem of Calculus

$$\int_a^b f(g(x))g'(x) dx = F(g(x)) \Big|_a^b = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(u) du.$$

□

Example 19.2. Evaluate $\int_0^{\pi/8} \sin^5 2x \cos 2x dx$.

Proof. Let $u = \sin 2x$ so that $du = 2 \cos 2x dx$. With this substitution,

- if $x = 0$, $u = \sin(0) = 0$,

$$\text{- if } x = \frac{\pi}{8}, u = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

So,

$$\int_0^{\pi/8} \sin^5 2x \cos 2x dx = \frac{1}{2} \int_0^{1/\sqrt{2}} u^5 du = \frac{1}{2} \frac{u^6}{6} \Big|_0^{1/\sqrt{2}} = \frac{1}{2} \left[\frac{1}{6(\sqrt{2})^6} - 0 \right] = \frac{1}{96}$$

□

Example 19.3. Evaluate $\int_0^{\ln 3} e^x (1 + e^x)^{1/2} dx$.

Proof. Making the u -substitution $u = 1 + e^x$ so that $du = e^x dx$, and changing the x -limits of integration ($x = 0, x = \ln 3$) to the u -limits $u = 1 + e^0 = 2, u = 1 + e^{\ln 3} = 1 + 3 = 4$. This yields

$$\int_0^{\ln 3} e^x (1 + e^x)^{1/2} dx = \int_2^4 u^{1/2} du = \frac{2}{3} u^{3/2} \Big|_2^4 = \frac{2}{3} [4^{3/2} - 2^{3/2}] = \frac{16 - 4\sqrt{2}}{3}.$$

□

19.1.1 Fundtions in the limits of integrals

There arise some situations where we want to evaluate integrals involving functions in their limits, that is, we want to evaluate $\int_a^{g(x)} f(t) dt$. In order to evaluate this integrals we let

$$F(x) = \int_a^x f(t) dt.$$

Then,

$$\int_a^{g(x)} f(t) dt = \frac{d}{dx} [F(g(x))] = F'(g(x)) g'(x) = f(g(x)) g'(x).$$

Therefore,

$$\frac{d}{dx} \left[\int_a^{g(x)} f(t) dt \right] = f(g(x)) g'(x).$$

Example 19.4.

$$\frac{d}{dx} \left[\int_1^{\sin x} (1 - t^2) dt \right] = (1 - \sin^2 x) \cos x = \cos^3 x.$$

Example 19.5.

$$\frac{d}{dx} \left[\int_0^{e^{-x}} \frac{1}{1 + t^4} dt \right] = \frac{1}{1 + (e^{-x})^4} \frac{d}{dx} [e^{-x}] = \frac{-e^{-x}}{1 + e^{-4x}}.$$

19.2 Integration by parts

Theorem 19.2. *If f and g are differentiable functions, then*

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

Proof. As

$$\frac{d}{dx} [f(x)g(x)] = f(x)g'(x) + g(x)f'(x).$$

Therefore,

$$\int [f(x)g'(x) + g(x)f'(x)] dx = f(x)g(x)$$

or

$$\int f(x)g'(x)dx + \int g(x)f'(x)dx = f(x)g(x)$$

We conclude by rearranging the terms. □

Remark 19.3. *Let $u = f(x)$ and $dv = g'(x)dx$. Then, we have $du = f'(x)dx$ and $v = g(x)$. Thus, by substitution, the formula for integration by parts becomes:*

$$\int u dv = uv - \int v du.$$

Example 19.6. Find $\int x \sin x dx$.

Solution. Suppose we choose $f(x) = x$, $g'(x) = \sin x$ then $f'(x) = 1$, $g(x) = -\cos x$. For g , we can choose any antiderivative of g' . Then,

$$\begin{aligned} \int x \sin x dx &= f(x)g(x) - \int g(x)f'(x)dx = x(-\cos x) - \int (-\cos x)dx \\ &= -x \cos x + \int \cos x dx = -x \cos x + \sin x + C. \end{aligned}$$

Alternatively, if we choose $u = f(x) = x$ and $dv = g'(x)dx = \sin x dx$. Therefore,

$$du = dx \quad \text{and} \quad v = -\cos x.$$

$$\begin{aligned} \int x \sin x dx &= \int x \sin x dx = x(-\cos x) - \int (-\cos x)dx \\ &= -x \cos x + \int \cos x dx = -x \cos x + \sin x + C. \end{aligned}$$

□

Remark 19.4. *Our aim in using integration by parts is to obtain a simpler integral than the one we started with.*

Thus, in previous example, we started with $\int x \sin x dx$ and expressed it in terms of the simpler integral $\int \cos x dx$. If we had instead chosen $u = \sin x$ and $dv = x dx$, then $du = \cos x dx$ and $v = \frac{x^2}{2}$. So, integration by parts gives:

$$\int x \sin x dx = (\sin x) \frac{x^2}{2} - \frac{1}{2} \int_x^2 \cos x dx$$

Although this is true, $\int x^2 \cos x dx$ is a more difficult integral than the one we started with.

Hence, when choosing f and dv , we usually try to keep $u = f(x)$ to be a function that becomes simpler when differentiated. At least, it should not be more complicated.

Example 19.7. Evaluate $\int \ln x dx$.

Solution. Here, we don't have much choice for u and dv . Let $u = \ln x$ and $dv = dx$. Then, $du = \frac{1}{x} dx$, and $v = x$. Integrating by parts, we get:

$$\int \ln x dx = x \ln x - \int x \frac{dx}{x} = x \ln x - \int dx = x \ln x - x + C.$$

□

Example 19.8. Calculate $\int_0^1 \tan^{-1} x dx$.

Solution. Let $u = \tan^{-1} x$ and $dv = dx$, then $du = \frac{dx}{1+x^2}$ and $v = x$

$$\begin{aligned} \int_0^1 \tan^{-1} x dx &= x \tan^{-1} x \Big|_0^1 - \int_0^1 \frac{x}{1+x^2} dx \\ &= 1 \cdot \tan^{-1} 1 - 0 \cdot \tan^{-1} 0 - \int_0^1 \frac{x}{1+x^2} dx = \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2} dx \end{aligned}$$

To evaluate this integral, we use the substitution $t = 1 + x^2$. Then, $dt = 2x dx$. So, when $x = 0$, $t = 1$ and when $x = 1$, $t = 2$. Hence,

$$\int_0^1 \frac{x}{1+x^2} dx = \frac{1}{2} \int_1^2 \frac{dt}{t} = \frac{1}{2} \ln |t| \Big|_1^2 = \frac{1}{2} (\ln 2 - \ln 1) = \frac{1}{2} \ln 2$$

Therefore,

$$\int_0^1 \tan^{-1} x dx = \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2} dx = \frac{\pi}{4} - \frac{\ln 2}{2}.$$

□

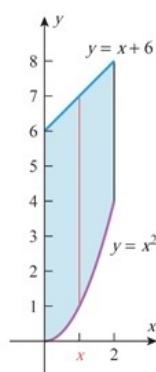


Figure 19.1: Area of the region bounded above by $y = x + 6$, bounded below by $y = x^2$, and bounded on the sides by the lines $x = 0$ and $x = 2$.

19.3 Area bounded by curves

Definition 19.5 (Area Formula). *If f and g are continuous functions on the interval $[a, b]$, and if $f(x) \geq g(x)$ for all x in $[a, b]$, then the area of the region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$ is*

$$A = \int_a^b [f(x) - g(x)] dx.$$

Example 19.9. *Find the area of the region bounded above by $y = x + 6$, bounded below by $y = x^2$, and bounded on the sides by the lines $x = 0$ and $x = 2$.*

Solution. The region and a cross section are shown in Figure 19.1. The cross section extends from $g(x) = x^2$ on the bottom to $f(x) = x + 6$ on the top. If the cross section is moved through the region, then its leftmost position will be $x = 0$ and its rightmost position will be $x = 2$. Thus,

$$A = \int_0^2 [(x + 6) - x^2] dx = \left[\frac{x^2}{2} + 6x - \frac{x^3}{3} \right]_0^2 = \frac{34}{3} - 0 = \frac{34}{3}.$$

□

Example 19.10. *Find the area of the region that is enclosed between the curves $y = x^2$ and $y = x + 6$.*

Solution. A sketch of the region in Figure 19.2 shows that the lower boundary is $y = x^2$ and the upper boundary is $y = x + 6$. At the endpoints of the region, the upper and the lower boundaries have the same y -coordinates; thus, to find the endpoints we equate

$$y = x^2 \quad \text{and} \quad y = x + 6.$$

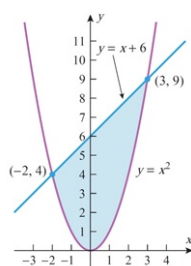


Figure 19.2: Area of the region that is enclosed between the curves $y = x^2$ and $y = x + 6$.

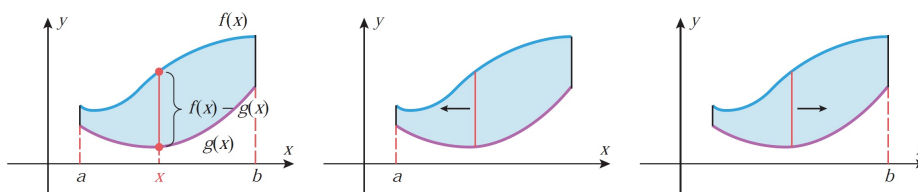


Figure 19.3: Finding the limits of integration for the area between two curves.

This yields $x^2 = x + 6 \Rightarrow x^2 - x - 6 = 0 \Rightarrow (x + 2)(x - 3) = 0$ from which we obtain $x = -2$ and $x = 3$.

In this case

$$A = \int_{-2}^3 [(x + 6) - x^2] dx = \left[\frac{x^2}{2} + 6x - \frac{x^3}{3} \right]_{-2}^3 = \frac{27}{2} - \left(-\frac{22}{3} \right) = \frac{125}{6}.$$

□

19.3.1 Finding the limits of integration for the area between two curves

1. Sketch the region and then draw a vertical line segment through the region at an arbitrary point x on the x -axis, connecting the top and bottom boundaries (see Figure 19.3 (left)).
2. The y -coordinate of the top endpoint of the line segment sketched in *Step 1* will be $f(x)$, the bottom one $g(x)$, and the length of the line segment will be $f(x) - g(x)$. This is the integrand.
3. To determine the limits of integration, imagine moving the line segment left and then right. The leftmost position at which the line segment intersects the region is $x = a$ and the rightmost is $x = b$ (see Figures 19.3 (center) and 19.3 (right)).